

A study of higher dimensional inhomogeneous cosmological model

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In this paper we present a class of exact inhomogeneous solutions to Einstein's equations for higher dimensional Szekeres metric with perfect fluid and a cosmological constant. We also show particular solutions depending on the choices of various parameters involved and for dust case. Finally, we examine the asymptotic behaviour of some of these solutions.

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I. INTRODUCTION

Usually, cosmological solutions to Einstein's field equations are obtained by imposing symmetries [1] on the space-time. One of the reasonable assumptions (in an average sense) is the spatial homogeneity. But when we consider cosmological phenomena over galactic scale or in smaller scale (detailed structure of the black body radiation) then we should drop the assumption of homogeneity i.e., inhomogeneous solutions are useful.

Szekeres [2] in 1975 gave a class of inhomogeneous solutions representing irrotational dust for the metric of the form (known as Szekeres metric)

$$ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} (dx^2 + dy^2)$$

Subsequently, the solutions have been extended by Szafron [3] and Szafron and Wainwright [4] for perfect fluid and they studied asymptotic behaviour for different choice of the parameters involved. Later Barrow and Stein-Schabes [5] gave solutions for dust model with a cosmological constant and showed the validity of the Cosmic 'no-hair' Conjecture.

In this work, we find inhomogeneous solutions for $(n + 2)$ -dimensional Szekeres space-time with perfect fluid and a cosmological constant. The paper is organized as follows: The basic equations are presented in section II while the solutions have been written in section III. An asymptotic study of particular solutions are given in section IV. Finally the paper ends with a short discussion in section V.

II. BASIC EQUATIONS

The metric for the $(n + 2)$ -dimensional Szekeres space-time is in the form

$$ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} \sum_{i=1}^n dx_i^2 \quad (1)$$

where α and β are functions of all the $(n + 2)$ space-time variables i.e.,

$$\alpha = \alpha(t, r, x_1, \dots, x_n), \quad \beta = \beta(t, r, x_1, \dots, x_n).$$

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The Einstein equations for the perfect fluid with a cosmological constant is of the form

$$G_{\mu\nu} = \Lambda g_{ab} + (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \quad (2)$$

where ρ and p are energy density and isotropic pressure measured by an observer moving with the fluid, Λ is the cosmological constant and u_μ is the fluid flow four vector. Since $u = \partial/\partial t$, the flow lines are geodesics and the contracted Bianchi identities imply that pressure is a function of t only i.e., $p = p(t)$. As there is no restriction on the energy density so ρ is in general a function of all the $(n+2)$ variables i.e., $\rho = \rho(t, r, x_1, \dots, x_n)$ and hence no equation of state is imposed.

Now from the non-vanishing components of the field equations (2) for the above metric (1), we have

$$\begin{aligned} n\dot{\alpha}\dot{\beta} + \frac{1}{2}n(n-1)\dot{\beta}^2 - e^{-2\beta} \sum_{i=1}^n \left\{ \alpha_{x_i}^2 + \frac{1}{2}(n-1)(n-2)\beta_{x_i}^2 + (n-2)\alpha_{x_i}\beta_{x_i} + \alpha_{x_i x_i} \right. \\ \left. + (n-1)\beta_{x_i x_i} \right\} + e^{-2\alpha} \left\{ n\alpha'\beta' - \frac{1}{2}n(n+1)\beta'^2 - n\beta'' \right\} = \Lambda + \rho \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{1}{2}n(n+1)\dot{\beta}^2 + n\ddot{\beta} - \frac{1}{2}n(n-1)e^{-2\alpha}\beta'^2 - e^{-2\beta} \sum_{i=1}^n \left\{ \frac{1}{2}(n-1)(n-2)\beta_{x_i}^2 + (n-1)\beta_{x_i x_i} \right\} \\ = \Lambda - p \end{aligned} \quad (4)$$

$$\begin{aligned} \ddot{\alpha}^2 + \ddot{\alpha} + (n-1)\dot{\alpha}\dot{\beta} + \frac{1}{2}n(n-1)\dot{\beta}^2 + (n-1)\ddot{\beta} + e^{-2\alpha} \left\{ (n-1)\alpha'\beta' - \frac{1}{2}n(n-1)\beta'^2 - \right. \\ \left. (n-1)\beta'' \right\} - e^{-2\beta} \sum_{i \neq j=1}^n \left\{ \alpha_{x_j}^2 + \frac{1}{2}(n-2)(n-3)\beta_{x_j}^2 + \alpha_{x_j x_j} + (n-2)\beta_{x_j x_j} + (n-3)\alpha_{x_j}\beta_{x_j} \right\} \\ - e^{-2\beta} \left\{ (n-1)\alpha_{x_i}\beta_{x_i} + \frac{1}{2}(n-1)(n-2)\beta_{x_i}^2 \right\} = \Lambda - p \end{aligned} \quad (5)$$

$$\alpha_{x_j}(-\alpha_{x_i} + \beta_{x_i}) + \beta_{x_j}(\alpha_{x_i} + (n-2)\beta_{x_i}) - \alpha_{x_i x_j} - (n-2)\beta_{x_i x_j} = 0, \quad (i \neq j) \quad (6)$$

$$\dot{\alpha}\beta' - \dot{\beta}\beta' - \dot{\beta}' = 0 \quad (7)$$

$$-\dot{\alpha}\alpha_{x_i} + \dot{\beta}\beta_{x_i} - \dot{\alpha}_{x_i} - (n-1)\dot{\beta}_{x_i} = 0 \quad (8)$$

$$\alpha_{x_i}\beta' - \beta'_{x_i} = 0 \quad (9)$$

where dot, dash and subscript stands for partial differentiation with respect to t , r and the corresponding variables respectively (e.g., $\beta_{x_i} = \frac{\partial \beta}{\partial x_i}$) with $i, j = 1, 2, \dots, n$.

From equations (7) and (9) after differentiating with respect to x_i and t respectively, we have the integrability condition

$$\dot{\beta}_{x_i}\beta'^2 = 0, \quad i = 1, 2, \dots, n \quad (10)$$

Thus, if $\beta' \neq 0$ we must have $\dot{\beta}_{x_i} = 0$, $i = 1, 2, \dots, n$. In the following section we shall consider the following possibilities

$$(i) \beta' \neq 0, \quad (ii) \beta' = 0, \quad \dot{\beta}_{x_i} = 0, \quad i = 1, 2, \dots, n$$

to get solutions of the field equations.

III. SOLUTIONS TO THE FIELD EQUATIONS

In this section we shall solve the field equations (3)-(9) using the above restrictions separately.

Case I: $\beta' \neq 0$

Here due to the restrictions $\dot{\beta}_{x_i} = 0$, $i = 1, 2, \dots, n$ we have from the field equations (7) and (9), the form of the metric coefficient as

$$e^\beta = R(t, r) e^{\nu(r, x_1, \dots, x_n)} \quad (11)$$

and

$$e^\alpha = R' + R \nu' \quad (12)$$

Now substituting these forms for the metric coefficient in equation (4) we have the differential equations for R and ν as

$$R\ddot{R} + \frac{1}{2}(n-1)\dot{R}^2 + \frac{1}{n}(p(t) - \Lambda)R^2 = \frac{n-1}{2n}f(r) \quad (13)$$

and

$$e^{-2\nu} \sum_{i=1}^n \{(n-2)\nu_{x_i}^2 + 2\nu_{x_i x_i}\} = f(r) - n \quad (14)$$

with $f(r)$ as arbitrary function of r alone.

Equation (13) can be integrated once to have the first integral

$$\dot{R}^2 = \frac{2\Lambda}{n(n+1)}R^2 + f(r) + \frac{F(r)}{R^{n-1}} - \frac{2}{n} R^{1-n} \int p(t)R^n dR \quad (15)$$

where $F(r)$ is another arbitrary function of r (appears due to integration).

Also from (14) the solution for ν will be

$$e^{-\nu} = A(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n B_i(r)x_i + C(r) \quad (16)$$

with the restriction

$$\sum_{i=1}^n B_i^2 - 4AC = f(r) - 1 \quad (17)$$

for the arbitrary functions $A(r)$, $B_i(r)$, $i = 1, 2, \dots, n$ and $C(r)$.

Further, to solve R completely let us consider p as a polynomial in t as

$$p(t) = p_0 t^{-a} \quad (18)$$

(p_0 and a are positive constants) and we have the general solution for R as

$$R^{\frac{n+1}{2}} = \begin{cases} \sqrt{t} \left\{ C_1 J_\xi \left[\frac{2\sqrt{c}}{|a-2|} t^{-\frac{a-2}{2}} \right] + C_2 Y_\xi \left[\frac{2\sqrt{c}}{|a-2|} t^{-\frac{a-2}{2}} \right] \right\} \\ \sqrt{t} \left\{ C_1 J_\xi \left[\frac{2\sqrt{c}}{|a-2|} t^{-\frac{a-2}{2}} \right] + C_2 J_{-\xi} \left[\frac{2\sqrt{c}}{|a-2|} t^{-\frac{a-2}{2}} \right] \right\} \\ C_1 t^{q_1} + C_2 t^{1-q_1} \end{cases} \quad (19)$$

according as ξ is an integer, non-integer and $a = 2$. Here C_1 and C_2 are arbitrary functions of r and we have chosen

$$\xi = \frac{1}{a-2}, \quad c = \frac{(n+1)p_0}{2n}, \quad q_1 = \frac{1}{2}(1 + \sqrt{1 - 4c}).$$

It is to be noted that to derive the above solution we have chosen $\Lambda = 0 = f(r)$. However, for non-zero Λ (but $f(r) = 0$) the solution is possible only for $a = 0$ and 2 as

$$R^{\frac{n+1}{2}} = \begin{cases} C_1 \cos \left\{ t \sqrt{\frac{n+1}{2n}} (p_0 - \Lambda) \right\} + C_2 \sin \left\{ t \sqrt{\frac{n+1}{2n}} (p_0 - \Lambda) \right\}, & \text{when } a = 0 \\ \sqrt{t} \left\{ C_1 J_\zeta \left[-\frac{it\sqrt{\Lambda}}{\sqrt{2}} \sqrt{1 + \frac{1}{n}} \right] + C_2 Y_\zeta \left[-\frac{it\sqrt{\Lambda}}{\sqrt{2}} \sqrt{1 + \frac{1}{n}} \right] \right\}, & \text{when } a = 2 \end{cases} \quad (20)$$

$$\text{with } \zeta = \frac{1}{2} \sqrt{1 - \frac{2(n+1)p_0}{n}}.$$

Further, if we consider the dust model (i.e., $p(t) = 0$) then the above solution (19) simplifies to $R^{(n+1)/2} \propto t$. Hence for the usual 4D (i.e., $n = 2$) the scale factor R grows as $t^{2/3}$ as in the usual Friedmann model.

Now, the physical and kinematical parameters have the following expressions

$$\rho = \frac{n}{2} \frac{F' + (n+1)F\nu'}{R^n(R' + R\nu')} - \frac{p_0}{t^a} \quad (21)$$

$$\theta = \frac{R\dot{R}' + (n+1)R\dot{R}\nu' + n\dot{R}R'}{R(R' + R\nu')} \quad (22)$$

$$\sigma^2 = \frac{n}{8(n+1)(n-1)^2} \left[\frac{2R^{n-1}(Rf' - 2R'f) + (n-1)(RF' - (n+1)R'F)}{\dot{R}R^n(R' + R\nu')} \right]^2 \quad (23)$$

Case II: $\beta' = \dot{\beta}_{x_i} = 0, \quad i = 1, 2, \dots, n$

In this case from the field equations we have the form of the metric functions

$$e^\beta = R(t) e^{\nu(x_1, x_2, \dots, x_n)} \quad (24)$$

and

$$e^\alpha = R(t) \eta(r, x_1, x_2, \dots, x_n) + \mu(t, r) \quad (25)$$

Then as before from the field equation (4) we have similar differential equations in R and ν as

$$R\ddot{R} + \frac{1}{2}(n-1)\dot{R}^2 + \frac{1}{n}(p(t) - \Lambda)R^2 = \frac{n-1}{2n}K \quad (26)$$

and

$$e^{-2\nu} \sum_{i=1}^n \{(n-2)\nu_{x_i}^2 + 2\nu_{x_i x_i}\} = K - n \quad (27)$$

with K , an arbitrary constant.

Here we take the solution for ν in the form

$$e^{-\nu} = P \sum_{i=1}^n x_i^2 + \sum_{i=1}^n Q_i x_i + S \quad (28)$$

where the arbitrary constants P, Q_i ($i = 1, 2, \dots, n$) and S are restricted as before

$$\sum_{i=1}^n Q_i^2 - 4PS = K - 1 \quad (29)$$

Now to determine the function η , we have from the field equation (6)

$$\frac{\partial^2(e^{-\nu}\eta)}{\partial x_i \partial x_i} = 0 \quad (30)$$

and then from the field equation (5) we have the solution

$$e^{-\nu}\eta = u(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n v_i(r)x_i + w(r) \quad (31)$$

with $u(r), v_i(r)$ ($i = 1, 2, \dots, n$) and $w(r)$ as arbitrary functions.

Further, to obtain the function μ we use the following combination of the field equations namely,

$$\sum_{i=1}^n G_{x_i}^{x_i} - G_r^r = (n-1)(\Lambda - p(t)) \quad (32)$$

and the resulting differential equation in μ is

$$R\ddot{\mu} + (n-1)\dot{R}\dot{\mu} + \mu \left[\ddot{R} + \frac{2}{n} (p(t) - \Lambda)R \right] = g(r) \quad (33)$$

with

$$g(r) = (n-1) \left[2(uS + wP) - \sum_{i=1}^n v_i Q_i \right] \quad (34)$$

For explicit solution if we choose $p(t)$ as in the previous case (see eq.(18)) then the explicit form for R is same as in equation (19) except here C_1 and C_2 are arbitrary constants and

we have chosen $K = \Lambda = 0$. Similarly, we have the same solutions (20) for non-zero Λ . But we note that the differential equation (33) is not solvable for any value of n . In fact only for $n = 3$ (i.e., for five dimension) we have the complete solution

$$\mu R = d_1 t^{q_2} + d_2 t^{1-q_2} \quad (35)$$

for $a = 2$ and $q_2 = \frac{1}{2} \left(1 + \sqrt{1 - \frac{8p_0}{3}} \right)$ with $\Lambda = g(r) = 0$.

Further, the physical and kinematical parameters have the expressions as

$$\rho = \frac{2n\dot{\mu}R - K}{2\mu R} + \frac{n}{n-1} \left[\frac{\eta(R^2\ddot{\mu} - n\mu R\ddot{R}) - (n-1)\mu^2\ddot{R}}{\mu R(\mu + \eta R)} \right] - p(t) \quad (36)$$

$$\theta = \frac{R\dot{\mu} + n\mu\dot{R} + (n+1)R\dot{R}\eta}{R(\mu + \eta R)} \quad (37)$$

$$\sigma^2 = \frac{n}{2(n+1)} \left[\frac{R\dot{\mu} - \dot{R}\mu}{R(\mu + \eta R)} \right]^2 \quad (38)$$

Finally, for simple dust case we have the solution for $\Lambda = 0$ in terms of hypergeometric function as

$$\sqrt{C_1}(t - t_0) = R^{\frac{n+1}{2}} {}_2F_1\left[\frac{1}{2}, \frac{n+1}{2n-2}, \frac{3n-1}{2n-2}, -\frac{(n+1)^2}{4nC_1} f(r) R^{n-1}\right] \quad (39)$$

But for non-zero Λ we can have solution only for $n = 3$ as

$$R^2 = C_1 e^{t\sqrt{\frac{2\Lambda}{3}}} + C_2 e^{-t\sqrt{\frac{2\Lambda}{3}}} - \frac{f(r)}{\Lambda} \quad (40)$$

However, we can have an integral equation from (15) (with $p = 0$) as

$$t - t_0 = \int \frac{dR}{\sqrt{\frac{2\Lambda}{n(n+1)} R^2 + f(r) + \frac{F(r)}{R^{n-1}}}} \quad (41)$$

and we have the following particular solutions :

(i) $F(r) = 0$

$$R = \sqrt{\frac{(n+1)f(r)}{2\Lambda}} \operatorname{Sinh} \left[(t - t_0) \sqrt{\frac{2\Lambda}{n(n+1)}} \right]$$

(ii) $f(r) = 0$ (i.e., $K = 0$ in *Case II*)

$$R^{\frac{n+1}{2}} = \sqrt{\frac{(n+1)F(r)}{2\Lambda}} \operatorname{Sinh} \left[(t - t_0) \sqrt{\frac{(n+1)\Lambda}{2n}} \right]$$

(iii) $\Lambda = 0$ ($n = 3$)

$$R^2 = \frac{1}{3}f(r)(t-t_0)^2 - \frac{3F(r)}{f(r)}$$

$$(iv) \quad f(r) = \Lambda = 0$$

$$R^{\frac{n+1}{2}} = \frac{n+1}{2} \sqrt{F(r)} (t-t_0)$$

$$(v) \quad f(r) = F(r) = 0$$

$$R = e^{(t-t_0)\sqrt{\frac{2\Lambda}{n(n+1)}}}$$

Also for the dust μ has the solution

$$R = \mu = \left[C_1 e^{t\sqrt{\frac{2\Lambda}{3}}} + C_2 e^{-t\sqrt{\frac{2\Lambda}{3}}} - \frac{K}{\Lambda} \right]^{\frac{1}{2}} \quad (42)$$

for $n = 3$.

IV. ASYMPTOTIC BEHAVIOUR

We shall now discuss the asymptotic behaviour of the solutions presented in the previous section for both perfect fluid and dust model separately. The co-ordinates vary over the range : $t_0 < t < \infty$; $-\infty < r < \infty$; $-\infty < x_i < \infty$, $i = 1, 2, \dots, n$.

A. Perfect fluid model

As $p \geq 0$, $p \neq 0$ so we must have $\frac{1}{2} < q_1 < 1$. We shall first consider the case when $a = 2$. For large t (*Case I* i.e., $\beta' \neq 0$)

$$R^{\frac{n+1}{2}} \sim t^{q_1}$$

$$\rho \sim \frac{n}{2}(F' + (n+1)F\nu')t^{-2q_1}$$

$$p \sim t^{-2}$$

$$\theta \sim t^{-1}$$

$$\sigma^2 \sim t^{-2}$$

For *Case II*, we have similar behaviour for large t together with $\mu \sim t^{\tilde{q}_1}$ where \tilde{q}_1 is the value of q_1 for $n = 3$ and also we have $\frac{1}{2} < \tilde{q}_1 < 1$. Thus as $t \rightarrow \infty$, (p, ρ) fall off faster compare to (θ, σ) , while the scale factor R (and μ) gradually increases with time. So the model approaches isotropy along fluid world line as $t \rightarrow \infty$.

B. Dust model

For the dust case with non-zero Λ (and $n = 3$) we have for large t ,

$$\mu = R \sim e^{t\sqrt{\frac{\Lambda}{6}}}$$

$$\rho = \rho_0, \quad a \text{ constant}$$

$$\theta = \theta_0, \quad a \text{ constant}$$

$$\sigma = \sigma_0, \quad a \text{ constant}$$

We note that for *Case I*, $\rho_0 = \sigma_0 = 0$ while for *Case II* we have $\rho_0 \neq 0$, $\sigma_0 \neq 0$. Thus universe will behave locally like de-sitter model in *Case I* though the global geometry will be different. However, in *Case II* the universe will not isotropize and for large t the measure of anisotropy and expansion scalar become finite constant.

V. CONCLUDING REMARKS

In this paper, we have found cosmological solutions for $(n + 2)$ -dimensional Szekeres form of metric with perfect fluid (or dust) as the matter distribution. We can classify the solutions in two categories namely, (i) $\beta' \neq 0$ and (ii) $\beta' = 0$. The first set of solutions are known as quasi-spherical solution while second class of solutions are termed as cylindrical type of solutions.

However, if we assume the arbitrary functions $A(r)$, $B_i(r)$ and $C(r)$ to have the constant values namely $A(r) = C(r) = \frac{1}{2}$ and $B_i(r) = 0$ (for all $i = 1, 2, \dots, n$) i.e., if we have chosen the arbitrary function $f(r)$ to be zero then using the transformation

$$\begin{aligned} x_1 &= \text{Sin}\theta_n \text{Sin}\theta_{n-1} \dots \text{Sin}\theta_2 \text{Cot}\frac{1}{2}\theta_1 \\ x_2 &= \text{Cos}\theta_n \text{Sin}\theta_{n-1} \dots \text{Sin}\theta_2 \text{Cot}\frac{1}{2}\theta_1 \\ x_3 &= \text{Cos}\theta_{n-1} \text{Sin}\theta_{n-2} \dots \text{Sin}\theta_2 \text{Cot}\frac{1}{2}\theta_1 \\ &\dots \dots \dots \dots \dots \dots \dots \dots \\ x_{n-1} &= \text{Cos}\theta_3 \text{Sin}\theta_2 \text{Cot}\frac{1}{2}\theta_1 \\ x_n &= \text{Cos}\theta_2 \text{Cot}\frac{1}{2}\theta_1 \end{aligned}$$

the Szekeres metric (1) with the solution (11), (12), (16) reduces to the $(n + 2)$ -dimensional spherically symmetric metric

$$ds^2 = dt^2 - R'^2 dr^2 - R^2 d\Omega_n^2$$

It is to be noted that if the arbitrary functions A, B_i, C depend on r , then we can not get the above spherical form of the space-time. So these arbitrary functions play an important role to identify the nature of the space-time.

Finally, the study of asymptotic behaviour shows that some of the solutions will become isotropic at late time while there are solutions for which shear scalar remains constant throughout the evolution. Thus Cosmic ‘no-hair’ Conjecture is not valid for all solutions. This violation of Cosmic ‘no-hair’ Conjecture is not unusual because the Szekeres metric

may not have always non-positive 3-space curvature scalar. Also in this higher dimensional Szekeres space-time we have solutions which expands as de Sitter in some directions but not in other directions.

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